TRANSLATING SOLITONS TO SYMPLECTIC AND LAGRANGIAN MEAN CURVATURE FLOWS

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Abstract. In this paper, we construct finite blow-up examples for symplectic mean curvature flows and we study properties of symplectic translating solitons. We prove that, the Kähler angle α of a symplectic translating soliton with $\max|A|=1$ satisfies that $\sup|\alpha| > \frac{\pi}{4}\frac{|T|}{|T|+1}$ where T is the direction in which the surface translates.

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1. Introduction

We consider the evolution of a symplectic surface (Lagrangian surface) in a Kähler-Einstein surface by its mean curvature, which we call a symplectic mean curvature flow (Lagrangian mean curvature flow). We [15] showed that, if the scalar curvature of the Kähler-Einstein surface is positive and the initial surface is sufficiently close to a holomorphic curve, then the symplectic mean curvature flow exists globally and converges to a holomorphic curve at infinity. In this paper, we construct examples to show that, in general the symplectic mean curvature flow may blow-up at a finite time. Therefore, it is necessary to study the singularity of a symplectic mean curvature flow.

Chen-Li ([3], [4], [5]) and Wang [24] independently proved that there is no Type I singularity along the symplectic (almost calibrated Lagrangian) mean curvature flow. Therefore, the structure of the type II singularity has been of great interest.

One of the most important examples of Type II singularity is the translating soliton (c.f. [11], [16]), which is one class of eternal solutions defined for $-\infty < t < \infty$. More precisely, the translating solitons are surfaces which evolve by translating in space with a constant velocity. Hamilton studied this kind of eternal solutions to the mean curvature flow of a hypersurface in $\mathbf{R}^n$ [11] and to the Ricci flow [12], [13]. The main purpose of this paper is to study the properties of symplectic translating solitons.

Definition *The translating soliton is called a standard translating soliton if the norm of its second fundamental form A satisfies that* $\max|A| = 1$. *Let*

$$ST = \{\Sigma \mid \Sigma \text{ is a standard symplectic translating soliton.}\}$$

Key words and phrases. Symplectic surface, Lagrangian surface, translating slotion, mean curvature flow.

Remark *Since we are interested in the translating solitons which arise in the blow-up analysis of singularities, it is reasonable to assume that* $\max|A| = 1$.

Main Theorem 1 *Suppose that* $\Sigma \in ST$ *is a symplectic standard translating soliton with Kähler angle* α*, then* $\sup_\Sigma|\alpha| > \frac{\pi}{4}\frac{|T|}{|T|+1}$*, where* T *is the direction in which the surface translates.*

Note that, the "grim reaper" $(x, y, -\ln\cos x, 0)$, $|x| < \pi/2$, $y \in \mathbf{R}$ is one standard symplectic translating soliton which translates in the direction of the constant vector $(0, 0, 1, 0)$. In this case the Kähler angle $\alpha = x$ and it is clear that $\sup_\Sigma|\alpha| = \pi/2$, i.e, $\inf_\Sigma\cos\alpha = 0$. In the study of symplectic mean curvature flows of compact surfaces, we assume that $\cos\alpha \geq \delta > 0$ on the initial surface and consequently [3] at each time t, on Σ_t, $\cos\alpha \geq \delta > 0$. One of the purpose of studying symplectic translating solitons is to rule out them as the limiting flows in the blow-up analysis at a type II singularity. More precisely, we believe,

Conjecture 1 A symplectic translating soliton can not be a limiting flow of rescaled surfaces at a type II singular point.

Even more we conjecture that

Conjecture 2 *Any blowing up at a type II singularity of a symplectic mean curvature flow is a union of non-flat minimal surfaces in* $\mathbb{R}^4$.

As an analogous result of translating solitons in almost calibrated mean curvature flow [19], we have,

Main Theorem 2 *Suppose that* Σ *is a standard translating soliton to the almost calibrated Lagrangian mean curvature flow with Lagrangian angle* θ*, then* $\sup_\Sigma|\theta| > \frac{\sqrt{2}\pi}{4}\frac{|T|}{|T|+1}$*, where* T *is the direction in which the surface translates.*

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2. Preliminaries

In this section, we fix some notations and recall some basic facts on symplectic mean curvature flows and Lagrangian mean curvature flows. We consider immersions

$$F_0 : \Sigma \to \mathbf{R}^4$$

of smooth surface Σ in $\mathbf{R}^4$. If Σ evolves along the mean curvature, then there is a one-parameter family $F_t = F(\cdot, t)$ of immersions which satisfy the mean curvature

flow equation:

$$\begin{cases} \frac{d}{dt}F(x,t) &= H(x,t) \\ F(x,0) &= F_0(x). \end{cases} \tag{2.1}$$

Here $H(x,t)$ is the mean curvature vector of $\Sigma_t = F_t(\Sigma)$ at $F(x,t)$.

Let ω and $\langle \cdot, \cdot \rangle$ denote the standard Kähler form and Euclidean metric on $\mathbf{R}^4$ respectively. We choose a local field of orthonormal frames e_1, e_2, v_1, v_2 of $\mathbf{R}^4$ along Σ such that e_1, e_2 are tangent vectors of Σ and v_1, v_2 are in the normal bundle over Σ. Denote the induced metric on Σ by (g_{ij}). The second fundamental form A and the mean curvature vector H of Σ can be expressed, in the local frame, as $A = A^\alpha v_\alpha$, and $H = -H^\alpha v_\alpha$, where and throughout this paper all repeated indices are summed over suitable range. For each α, the coefficient A^α is a 2×2 matrix $(h^\alpha_{ij})_{2\times 2}$. By the Weingarten equation (cf. [22]), we have

$$\begin{aligned} h^\alpha_{ij} &= \langle v_\alpha, \overline{\nabla}_i \overline{\nabla}_j F\rangle = -\langle \overline{\nabla}_j v_\alpha, \overline{\nabla}_i F\rangle = h^\alpha_{ji}, \\ H^\alpha &= g^{ij} h^\alpha_{ij} = h^\alpha_{ii}, \end{aligned}$$

where $\overline{\nabla}$ is the connection on $\mathbf{R}^4$. The norm of the second fundamental form of Σ is:

$$|A|^2 = \sum_\alpha |A^\alpha|^2 = g^{ij} g^{kl} h^\alpha_{ik} h^\alpha_{jl} = h^\alpha_{ik} h^\alpha_{ik}.$$

A surface Σ is called *symplectic* if the Kähler angle α (c.f. [9]) satisfies that $\cos\alpha > 0$, where $\cos\alpha$ is defined by $\omega|_\Sigma = \cos\alpha d\mu_\Sigma$, where $d\mu_\Sigma$ is the induced volume form of Σ, and ω is the standard Kähler form on $\mathbf{C}^2$.

Suppose Σ is symplectic and evolves along the mean curvature in $\mathbf{R}^4$ which is called symplectic mean curvature flow. Let J_{Σ_t} be an almost complex structure in a tubular neighborhood of Σ in $\mathbf{R}^4$ with

$$\begin{cases} J_{\Sigma_t} e_1 &= e_2 \\ J_{\Sigma_t} e_2 &= -e_1 \\ J_{\Sigma_t} v_1 &= v_2 \\ J_{\Sigma_t} v_2 &= -v_1. \end{cases} \tag{2.2}$$

It is not difficult to verify ([6] and [3]) that,

$$\begin{aligned} |\overline{\nabla} J_{\Sigma_t}|^2 &= |h^2_{11} + h^1_{12}|^2 + |h^2_{21} + h^1_{22}|^2 + |h^2_{12} - h^1_{11}|^2 + |h^2_{22} - h^1_{21}|^2 \\ &= \frac{1}{2}|H|^2 + \frac{1}{2}\left(((h^1_{11} + h^1_{22}) - 2(h^2_{12} + h^1_{22}))^2 + ((h^2_{11} + h^2_{22}) - 2(h^1_{21} + h^2_{11}))^2\right) \\ &\geq \frac{1}{2}|H|^2. \end{aligned}$$

Recall that ([3]) the Kähler angle α of Σ_t in $\mathbf{R}^4$ satisfies the parabolic equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\overline{\nabla} J_{\Sigma_t}|^2 \cos\alpha. \tag{2.3}$$

Suppose that the initial surface is symplectic, i.e., $\cos\alpha > 0$, then by applying the parabolic maximum principle to this evolution equation, one concludes that $\cos\alpha$

remains positive as long as the mean curvature flow has a smooth solution (cf. [7], [3], [24]).

Let J denote the standard complex structure on $\mathbf{C}^2$. We also consider a parallel holomorphic $(2,0)$ form,

$$\Omega = dz_1 \wedge dz_2.$$

A surface Σ is said to be *Lagrangian* if $\omega|_\Sigma = 0$. This implies that (see [14])

$$\Omega|_\Sigma = e^{i\theta} d\mu_\Sigma,$$

where $d\mu_\Sigma$ denotes the induced volume form of Σ and θ is called Lagrangian angle which is some multivalued function. If $\cos\theta > 0$, then Σ is called *almost-calibrated*. The relation between the Lagrangian angle and the mean curvature vector is given in [14] (also see [23]),

$$H = J\nabla\theta.$$

Suppose Σ is Lagrangian and evolves by the mean curvature, Smoczyk has shown that ([19], [20], [21]),

$$(\frac{\partial}{\partial t} - \Delta)\cos\theta = |H|^2\cos\theta. \tag{2.4}$$

If the initial surface is almost calibrated, then for every t, Σ_t is also almost calibrated, i.e. $\cos\theta > 0$, along the mean curvature flow by the parabolic maximum principle.

3. Property of translating solitons

Suppose that Σ_t is a translating soliton which translates in the direction of the constant vector T. That means $F_t = F + tT$, i.e, $\Sigma_t = \Sigma + tT$. Let $V = v^i e_i$ be the tangent part of T. Then the normal component must be $N = H^\alpha v_\alpha$ to solve the mean curvature flow. If we take the equation $v^i e_i + H^\alpha v_\alpha = T$ and differentiate it, then we get

$$\overline{\nabla}_j v^i e_i + v^i h^\alpha_{ij} v_\alpha + \overline{\nabla}_j H^\alpha v_\alpha - H^\alpha h^\alpha_{ij} e_i = 0,$$

where we assume that $\nabla_{e_i} e_j = 0$ at the considered point. Separating the tangential and normal part we get that,

$$\begin{aligned} \overline{\nabla}_j v^i &= H^\alpha h^\alpha_{ij} \\ \overline{\nabla}_j H^\alpha &= -v^i h^\alpha_{ij}. \end{aligned} \tag{3.1}$$

Using these equations we can get the following identities for the translating solitons.

Proposition 3.1. *On the translating soliton, the Kähler angle and the Lagrangian angle satisfy the following elliptic equations,*

$$-\Delta\cos\alpha = |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + v^i\nabla_i\cos\alpha, \tag{3.2}$$

and

$$-\Delta\cos\theta = |H|^2\cos\theta + v^i\nabla_i\cos\theta \tag{3.3}$$

Proof. Along the translating soliton, we have $T(\cos\alpha) = 0$. It implies that $H^\alpha v_\alpha(\cos\alpha) = -v^i\nabla_i\cos\alpha$. In (2.3), we have $\frac{\partial}{\partial t}\cos\alpha = H^\alpha v_\alpha(\cos\alpha)$, which implies that

$$-\Delta\cos\alpha \;=\; |\overline{\nabla} J_{\Sigma_t}|^2\cos\alpha + v^i\nabla_i\cos\alpha.$$

Using (2.4), similarly we can show (3.3). Q. E. D.

By solving the system of equations,

$$\begin{aligned}\overline{\nabla}_j H^\alpha &= -v^i h^\alpha_{ij},\\ h^1_{11} + h^1_{22} &= H^1,\\ h^2_{11} + h^2_{22} &= H^2,\end{aligned}$$

we get that at the points $|V| \neq 0$,

$$\begin{aligned}h^1_{11} &= -(v^1\overline{\nabla}_1 H^1 - v^2\overline{\nabla}_2 H^1 - H^1(v^2)^2)/|V|^2,\\ h^1_{12} &= -(v^1\overline{\nabla}_2 H^1 + v^2\overline{\nabla}_1 H^1 + H^1 v^2 v^1)/|V|^2,\\ h^1_{22} &= (v^1\overline{\nabla}_1 H^1 - v^2\overline{\nabla}_2 H^1 + H^1(v^1)^2)/|V|^2,\\ h^2_{11} &= -(v^1\overline{\nabla}_1 H^2 - v^2\overline{\nabla}_2 H^2 - H^2(v^2)^2)/|V|^2,\\ h^2_{12} &= -(v^1\overline{\nabla}_2 H^2 + v^2\overline{\nabla}_1 H^2 + H^2 v^2 v^1)/|V|^2,\\ h^2_{22} &= (v^1\overline{\nabla}_1 H^2 - v^2\overline{\nabla}_2 H^2 + H^2(v^1)^2)/|V|^2.\end{aligned}$$

Then it is easy to check that,

Proposition 3.2. *On the translating soliton, at the points $|V| \neq 0$,*

$$|A|^2 = |H|^2 + 2\frac{|\nabla H|^2}{|V|^2} + \frac{v^i\nabla_i|H|^2}{|V|^2}.$$

Proposition 3.3. *On the translating soliton, at the points $|V| \neq 0$, the mean curvature vector satisfies that,*

$$\begin{aligned}\Delta|H|^2 &= 2|\nabla H|^2 - 2(H^\alpha h^\alpha_{ij})^2 - v^i\nabla_i|H|^2\\ &= 2|\nabla H|^2 - 2|H|^4 - v^i\nabla_i|H|^2 - \frac{|\nabla|H|^2|^2}{|V|^2} - 2\frac{|H|^2}{|V|^2}v^i\nabla_i|H|^2. \qquad (3.4)\end{aligned}$$

Proof. Using equation (3.1) again, we have,

$$\begin{aligned}\Delta|H|^2 &= 2H^\alpha\Delta H^\alpha + 2|\nabla H^\alpha|^2\\ &= -2H^\alpha\overline{\nabla}_j(v^i h^\alpha_{ij}) + 2|\nabla H|^2\\ &= -2H^\alpha v^i\overline{\nabla}_i H^\alpha - 2H^\alpha h^\alpha_{ij}\overline{\nabla}_j v^i + 2|\nabla H|^2\\ &= 2|\nabla H|^2 - v^i\nabla_i|H|^2 - 2(H^\alpha h^\alpha_{ij})^2.\end{aligned}$$

Putting h^α_{ij} into the above equation, we can prove the proposition. Q. E. D.

In the sequence we prove one monotonicity formula for translating solitons. Let $H(X, X_0, t_0, t)$ be the backward heat kernel on $\mathbf{R}^4$. Define

$$\begin{aligned}\rho(F,t) &= 4\pi(t_0-t)H(F,X_0,t_0,t)\\ &= \frac{1}{4\pi(t_0-t)}\exp(-\frac{|F-X_0|^2}{4(t_0-t)})\end{aligned}$$

for $t < t_0$.

Proposition 3.4. *On the translating soliton, the Kähler angle satisfies the following formula,*

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_{\Sigma_t}\frac{1}{\cos\alpha}\rho(F,X_0,t,t_0)d\mu_t\right)\\ &=-\left(\int_{\Sigma_t}\frac{1}{\cos\alpha}\rho(F,X_0,t,t_0)\left|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\right|^2 d\mu_t\right.\\ &+\int_{\Sigma_t}\frac{1}{\cos\alpha}\rho(F,X_0,t,t_0)\left|\overline{\nabla}J_{\Sigma_t}\right|^2 d\mu_t\\ &\left.+\int_{\Sigma_t}\frac{2}{\cos^3\alpha}\left|\nabla\cos\alpha\right|^2\rho(F,X_0,t,t_0)d\mu_t\right). \qquad (3.5)\end{aligned}$$

Proof. A straightforward calculation shows that,

$$\begin{aligned}\frac{\partial}{\partial t}\rho &= (\frac{1}{t_0-t}-\frac{\frac{\partial F}{\partial t}\cdot(F-X_0)}{2(t_0-t)}-\frac{|F-X_0|^2}{4(t_0-t)^2})\rho\\ &= (\frac{1}{t_0-t}-\frac{T\cdot(F-X_0)}{2(t_0-t)}-\frac{|F-X_0|^2}{4(t_0-t)^2})\rho\\ &= (\frac{1}{t_0-t}-\frac{H\cdot(F-X_0)}{2(t_0-t)}-\frac{V\cdot(F-X_0)}{2(t_0-t)}-\frac{|F-X_0|^2}{4(t_0-t)^2})\rho.\end{aligned}$$

On Σ_t we have,

$$\Delta\rho=(\frac{\langle F-X_0,\nabla F\rangle^2}{4(t_0-t)^2}-\frac{\langle F-X_0,\Delta F\rangle}{2(t_0-t)}-\frac{|\nabla F|^2}{2(t_0-t)})\rho,$$

where ∇, Δ are connection and Laplacian on Σ_t in the induced metric. Note that,

$$|\nabla F|^2=2,\ \ \Delta F=H,$$

and add these two equations together, we get,

$$(\frac{\partial}{\partial t}+\Delta)\rho=-\left(\left|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\right|^2-|H|^2+\frac{V\cdot(F-X_0)}{2(t_0-t)}\right)\rho.$$

Now we have,

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\Sigma_t}\frac{1}{\cos\alpha}\rho(F,X_0,t,t_0)d\mu_t\\ &=\int_{\Sigma_t}\frac{1}{\cos\alpha}(\frac{\partial}{\partial t}+\Delta)\rho d\mu_t-\int_{\Sigma}\frac{1}{\cos\alpha}\Delta\rho d\mu_t\end{aligned}$$

$$= -\int_{\Sigma_t} \frac{1}{\cos\alpha}\left(|H + \frac{(F-X_0)^\perp}{2(t_0-t)}|^2 - |H|^2 + \frac{V\cdot(F-X_0)}{2(t_0-t)}\right)\rho d\mu_t$$
$$-\int_{\Sigma_t} \Delta(\frac{1}{\cos\alpha})\rho d\mu_t.$$

Using equation (3.2), we get that,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho(F, X_0, t, t_0) d\mu_t \\
&= -\int_{\Sigma_t} \frac{1}{\cos\alpha}\left(|H + \frac{(F-X_0)^\perp}{2(t_0-t)}|^2 - |H|^2\right)\rho d\mu_t + \int_{\Sigma_t} \frac{1}{\cos\alpha} V(\rho) d\mu_t \\
&- \int_{\Sigma_t} (\frac{|\overline{\nabla} J_{\Sigma_t}|^2}{\cos\alpha} - V(\frac{1}{\cos\alpha}))\rho d\mu_t - \int_{\Sigma_t} \frac{2}{\cos^3\alpha}|\nabla\cos\alpha|^2\rho d\mu_t \\
&= -\int_{\Sigma_t} \frac{1}{\cos\alpha}\left(|H + \frac{(F-X_0)^\perp}{2(t_0-t)}|^2 - |H|^2\right)\rho d\mu_t \\
&- \int_{\Sigma_t} \frac{2}{\cos^3\alpha}|\nabla\cos\alpha|^2\rho d\mu_t - \int_{\Sigma_t} \frac{|\overline{\nabla} J_{\Sigma_t}|^2}{\cos\alpha}\rho d\mu_t \\
&+ \int_{\Sigma_t} V(\frac{1}{\cos\alpha}\rho d\mu_t) - \int_{\Sigma_t} \frac{1}{\cos\alpha}\rho V(d\mu_t) \\
&= -\int_{\Sigma_t} \frac{1}{\cos\alpha}\left(|H + \frac{(F-X_0)^\perp}{2(t_0-t)}|^2 - |H|^2\right)\rho d\mu_t \\
&- \int_{\Sigma_t} \frac{2}{\cos^3\alpha}|\nabla\cos\alpha|^2\rho d\mu_t - \int_{\Sigma_t} \frac{|\overline{\nabla} J_{\Sigma_t}|^2}{\cos\alpha}\rho d\mu_t \\
&- \int_{\Sigma_t} H(\frac{1}{\cos\alpha}\rho d\mu_t) + \int_{\Sigma_t} \frac{1}{\cos\alpha}\rho H(d\mu_t).
\end{aligned} \tag{3.6}$$

Recall that

$$H(d\mu_t) = \frac{\partial}{\partial t}(d\mu_t) = -|H|^2 d\mu_t,$$

thus,

$$\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho H(d\mu_t) = -\int_{\Sigma_t} \frac{|H|^2}{\cos\alpha}\rho d\mu_t.$$

Since H is normal to Σ_t, we have,

$$\begin{aligned}
\int_{\Sigma_t} H(\frac{1}{\cos\alpha}\rho d\mu_t) &= H(\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho d\mu_t) \\
&= -V(\int_{\Sigma_t} \frac{1}{\cos\alpha}\rho d\mu_t) = 0.
\end{aligned}$$

Adding these identities into (3.6), we can prove the proposition. Q. E. D.

By the same argument, we can also get one monotonicity formula for Lagrangian angle.

Proposition 3.5. *On the translating soliton, the Lagrangian angle satisfies the following formula,*

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,X_0,t,t_0)d\mu_t\right)\\
&= -\left(\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,X_0,t,t_0)\left|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\right|^2 d\mu_t\right.\\
&+\int_{\Sigma_t}\frac{1}{\cos\theta}\rho(F,X_0,t,t_0)\,|H|^2\,d\mu_t\\
&\left.+\int_{\Sigma_t}\frac{2}{\cos^3\theta}\,|\nabla\cos\theta|^2\,\phi\rho(F,X_0,t,t_0)d\mu_t\right).
\end{aligned}\tag{3.7}$$

4. Proof of Main theorems

Using the gradient estimates, we can prove our main theorem. First we give an estimate of the evolution equation of $|H|^2$. Recall that we assume $|A|^2\le 1$. Applying the equations in Proposition 3.3, we have that if $|H|^2\ge\varepsilon$, then

$$\begin{aligned}
\Delta|H|^2 &\ge 2|\nabla H|^2-2|H|^2-v^i\nabla_i|H|^2\\
&\ge 2|\nabla H|^2-\frac{2}{\varepsilon}|H|^4-v^i\nabla_i|H|^2,
\end{aligned}$$

and if $|H|^2<\varepsilon$, then by Cauchy-Schwartz inequality we have,

$$\begin{aligned}
\Delta|H|^2 &= 2|\nabla H|^2-2|H|^4-v^i\nabla_i|H|^2-\frac{|\nabla|H|^2|^2}{|V|^2}-2\frac{|H|^2}{|V|^2}v^i\nabla_i|H|^2\\
&\ge 2|\nabla H|^2-(2+\delta)|H|^4-v^i\nabla_i|H|^2-4(1+\frac{1}{\delta})\frac{\varepsilon}{|T|-\varepsilon}|\nabla|H||^2.
\end{aligned}$$

Set $\tilde{a}=\frac{2}{\varepsilon}=2+\delta$, $\tilde{b}=\frac{1}{2}+(1+\frac{1}{\delta})\frac{\varepsilon}{|T|-\varepsilon}$, then in any case we have,

$$\Delta|H|^2\ge(4-4\tilde{b})|\nabla|H||^2-\tilde{a}|H|^4-v^i\nabla_i|H|^2.\tag{4.1}$$

We first prove Main Theorem in Lagrangian case.

Main Theorem 2 *Suppose that Σ is a standard translating soliton to the almost calibrated Lagrangian mean curvature flow with Lagrangian angle θ, then $\sup_\Sigma|\theta|>\frac{\sqrt{2}\pi}{4}\frac{|T|}{|T|+1}$, where T is the direction in which the surface translates.*

Proof. We argue it by contradiction. Suppose there is a standard translating soliton with $\cos(\sqrt{ab}\theta)\ge\delta_0$ for some $\delta_0>0$, where a,b are constants which are determined later. Now we consider the function

$$f=\frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}.$$

It is well know that

$$-\Delta\theta=H^\alpha v^\alpha(\theta)=-v^i\nabla_i\theta.$$

Using (3.3) and (4.1), we can compute Δf,

$$\begin{aligned}
\Delta\frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)} &= \frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}\Delta|H|^2+|H|^2\Delta\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}+2\nabla|H|^2\cdot\nabla\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}\\
&\geq \cos^{-\frac{1}{b}}(\sqrt{ab}\theta)((4-4\tilde{b})|\nabla|H||^2-\tilde{a}|H|^4-v^i\nabla_i|H|^2)\\
&\quad -|H|^2v^i\nabla_i\cos^{-\frac{1}{b}}(\sqrt{ab}\theta)+(\frac{a}{b}+a)\cos^{-\frac{1}{b}}(\sqrt{ab}\theta)\tan^2(\sqrt{ab}\theta)|H|^4\\
&\quad +a\cos^{-\frac{1}{b}}(\sqrt{ab}\theta)|H|^4-2\frac{a}{b}\cos^{-\frac{1}{b}}(\sqrt{ab}\theta)\tan^2(\sqrt{ab}\theta)|H|^4\\
&\quad +2\cos^{\frac{1}{b}}(\sqrt{ab}\theta)\nabla f\cdot\nabla\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}\\
&\geq \cos^{-\frac{1}{b}}(\sqrt{ab}\theta)\Big[(4-4\tilde{b})|\nabla|H||^2-\tilde{a}|H|^4+(\frac{a}{b}+a)\tan^2\sqrt{ab}\theta|H|^4\\
&\quad +a|H|^4-2\frac{a}{b}\tan^2\sqrt{ab}\theta|H|^4\Big]\\
&\quad -v^i\nabla_i f+2\cos^{\frac{1}{b}}(\sqrt{ab}\theta)\nabla f\cdot\nabla\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)}.
\end{aligned}$$

Since f is bounded, and the second fundamental form of Σ is bounded which implies that the curvature of Σ is bounded, we can apply the generalized maximal principle ([8] Theorem 3) to conclude that there is a sequence $\{x_k\}$ in Σ, such that,

$$\lim_{k\to\infty} f(x_k)=\sup_{x\in\Sigma} f(x),$$

$$\lim_{k\to\infty}|\nabla f(x_k)|=0,$$

and

$$\lim_{k\to\infty}\triangle f(x_k)\leq 0.$$

It implies that, at x_k,

$$\nabla|H|^2=|H|^2\sqrt{\frac{a}{b}}\tan(\sqrt{ab}\theta)\nabla\theta+o_k(1),$$

where $o_k(1)\to 0$ as $k\to\infty$, thus we get that,

$$|\nabla|H||^2=\frac{a}{4b}\tan^2(\sqrt{ab}\theta)|H|^4+o_k(1).$$

Putting this identity into the above inequality, we obtain that, at point x_k,

$$\begin{aligned}
\Delta\frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\theta)} &\geq \cos^{-\frac{1}{b}}(\sqrt{ab}\theta)\Big[\frac{a}{b}(1-\tilde{b})\tan^2(\sqrt{ab}\theta)|H|^4+(a-\tilde{a})|H|^4\\
&\quad +(a-\frac{a}{b})\tan^2(\sqrt{ab}\theta)|H|^4\Big]+o_k(1)\\
&\geq \cos^{-\frac{1}{b}}(\sqrt{ab}\theta)\Big[(a-\frac{\tilde{b}}{b}a)\tan^2(\sqrt{ab}\theta)|H|^4+(a-\tilde{a})|H|^4\Big]+o_k(1).
\end{aligned}$$

Set $b = \tilde{b}$ and $a > \tilde{a}$. Then at point x_k,

$$(a - \tilde{a}) \cos^{-\frac{1}{b}}(\sqrt{ab}\theta)|H|^4 \leq o_k(1),$$

which implies that $\lim_{k\to\infty} |H|^2(x_k) = 0$. This is equivalent to $\lim_{k\to\infty} f(x_k) = 0$. That is not possible. Thus there is no translating soliton with $\cos(\sqrt{ab}\theta) \geq \delta_0 > 0$. This implies that $\inf_\Sigma \cos(\sqrt{ab}\theta) \leq 0$, i.e, $\sup_\Sigma \sqrt{ab}|\theta| \geq \frac{\pi}{2}$. Now we estimate ab. Since $\frac{2}{\varepsilon} = 2 + \delta$, so $\delta = \frac{2-2\varepsilon}{\varepsilon}$

$$\begin{aligned} ab > \tilde{a}\tilde{b} &= \frac{2}{\varepsilon}(\frac{1}{2} + (1 + \frac{1}{\delta})\frac{\varepsilon}{|T| - \varepsilon}) \\ &= \frac{2}{\varepsilon}(\frac{1}{2} + \frac{2-\varepsilon}{2-2\varepsilon}\frac{\varepsilon}{|T| - \varepsilon}) \\ &= \frac{1}{\varepsilon} + \frac{2-\varepsilon}{(1-\varepsilon)(|T| - \varepsilon)}. \end{aligned}$$

Choose $\varepsilon = \frac{|T|}{|T|+1}$, then $ab > 2\frac{(|T|+1)^2}{|T|^2}$. Therefore, $\sup_\Sigma |\theta| > \frac{\sqrt{2}\pi}{4}\frac{|T|}{|T|+1}$. This completes the proof.

Q. E. D.

Before proving the Main Theorem 1, we need the following lemma.

Lemma 4.1. *Suppose that Σ is a surface in $\mathbf{R}^4$, we have,*

$$|\nabla\alpha|^2 \leq |\overline{\nabla} J_\Sigma|^2,$$

at the points where α is smooth.

Proof. In fact we can choose the local orthonormal frame of $\{e_1, e_2, v_1, v_2\}$ on $\mathbf{R}^4$ along Σ so that ω takes the following form (cf. [3], [6], [9]),

$$\omega = \cos\alpha u_1 \wedge u_2 + \cos\alpha u_3 \wedge u_4 + \sin\alpha u_1 \wedge u_3 - \sin\alpha u_2 \wedge u_4$$

where $\{u_1, u_2, u_3, u_4\}$ is the dual frame of $\{e_1, e_2, v_1, v_2\}$, and

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}.$$

Then

$$\cos\alpha = \omega(e_1, e_2).$$

For the sake of simplicity, we can assume the covariant derivatives of the orthonormal frame $\{e_1, e_2, v_1, v_2\}$ satisfy

$$\nabla_{e_i} e_j = 0.$$

We see that,

$$\begin{aligned} \nabla_1 \cos\alpha &= \omega(\overline{\nabla}_1 e_1, e_2) + \omega(e_1, \overline{\nabla}_1 e_2) \\ &= h^\alpha_{11}\omega(v_\alpha, e_2) + h^\alpha_{12}\omega(e_1, v_\alpha) \\ &= (h^2_{11} + h^1_{12})\sin\alpha, \end{aligned}$$

and

$$\begin{aligned}\nabla_2 \cos\alpha &= \omega(\overline{\nabla}_2 e_1, e_2) + \omega(e_1, \overline{\nabla}_2 e_2)\\ &= h^\alpha_{21}\omega(v_\alpha, e_2) + h^\alpha_{22}\omega(e_1, v_\alpha)\\ &= (h^1_{22} + h^2_{12})\sin\alpha,\end{aligned}$$

Thus

$$|\nabla \cos\alpha|^2 = (|h^2_{11} + h^1_{12}|^2 + |h^1_{22} + h^2_{12}|^2)\sin^2\alpha$$

i.e,

$$\begin{aligned}|\nabla\alpha|^2 &= (|h^2_{11} + h^1_{12}|^2 + |h^1_{22} + h^2_{12}|^2)\\ &\le |\overline{\nabla} J_\Sigma|^2 = |h^2_{11} + h^1_{12}|^2 + |h^2_{21} + h^1_{22}|^2 + |h^2_{12} - h^1_{11}|^2 + |h^2_{22} - h^1_{21}|^2.\end{aligned}$$

Remark 4.2. *It is not hard to see that* $|\nabla\alpha|^2 = \frac{1}{2}|\overline{\nabla} J_\Sigma|^2$ *if* $|H|^2 = 0$.

Now we begin to prove Main Theorem 1.

Main Theorem 1 *Suppose that* $\Sigma \in ST$ *is a symplectic standard translating soliton with Kähler angle* α, *then* $\sup_\Sigma |\alpha| > \frac{\pi}{4}\frac{|T|}{|T|+1}$, *where* T *is the direction in which the surface translates.*

Proof. We also prove it by contradiction. Assume that there is a translating soliton with $\cos(\sqrt{ab}\alpha) \ge \delta_0 > 0$, where a, b are constants which are determined later. We consider the function

$$f = \frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)}.$$

From (3.2) we see that, at the point where α is smooth,

$$\sin\alpha\Delta\alpha = (|\overline{\nabla} J_\Sigma|^2 - |\nabla\alpha|^2)\cos\alpha + v^i\nabla_i\cos\alpha.$$

Then at the point where $\sin\alpha \ne 0$ (Note that α is smooth at the point where $\sin\alpha \ne 0$), we have,

$$\Delta\alpha = \frac{\cos\alpha}{\sin\alpha}(|\overline{\nabla} J_\Sigma|^2 - |\nabla\alpha|^2) - v^i\nabla_i\alpha.$$

Thus using (4.1) we obtain that,

$$\begin{aligned}\Delta\frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)} &= \frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)}\Delta|H|^2 + |H|^2\Delta\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)} + 2\nabla|H|^2\cdot\nabla\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)}\\ &\ge \cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)((4-4\tilde{b})|\nabla|H||^2 - \tilde{a}|H|^4 - v^i\nabla_i|H|^2)\\ &\quad -|H|^2v^i\nabla_i\cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)\\ &\quad +\sqrt{\frac{a}{b}}\cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)\cot\alpha\tan(\sqrt{ab}\alpha)(|\overline{\nabla} J_\Sigma|^2 - |\nabla\alpha|^2)|H|^2\\ &\quad +(\frac{a}{b} + a)\cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)\tan^2(\sqrt{ab}\alpha)|H|^2|\nabla\alpha|^2\\ &\quad +a\cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)|H|^2|\nabla\alpha|^2\\ &\quad -2\frac{a}{b}\cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)\tan^2(\sqrt{ab}\alpha)|H|^2|\nabla\alpha|^2\end{aligned}$$

$$+2\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)\nabla f\cdot\nabla\frac{1}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)}.$$

Since f is bounded, and the second fundamental form of Σ is bounded which implies that the curvature of Σ is bounded, we can apply the generalized maximal principle ([8] Theorem 3) to conclude that there is a sequence $\{x_k\}$ in Σ, such that,

$$\lim_{k\to\infty} f(x_k)=\sup_{x\in\Sigma} f(x),$$

$$\lim_{k\to\infty}|\nabla f(x_k)|=0,$$

and

$$\lim_{k\to\infty}\triangle f(x_k)\leq 0.$$

It implies that

$$|\nabla|H||^2=\frac{a}{4b}\tan^2(\sqrt{ab}\alpha)|\nabla\alpha|^2|H|^2+o_k(1),$$

where $o_k(1)\to 0$ as $k\to\infty$. By Lemma 4.1 and notice that $\cot\alpha\tan(\sqrt{ab}\alpha)\leq\sqrt{ab}$, then we get that at x_k,

$$\begin{aligned}
\Delta\frac{|H|^2}{\cos^{\frac{1}{b}}(\sqrt{ab}\alpha)} \quad\geq\quad & \cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)\Big[\frac{a}{b}(1-\tilde{b})\tan^2(\sqrt{ab}\theta)|\nabla\alpha|^2|H|^2\\
&+a(|\overline{\nabla}J_\Sigma|^2-|\nabla\alpha|^2)|H|^2\\
&-\tilde{a}|H|^4+a|\nabla\alpha|^2|H|^2\\
&(a-\frac{a}{b})\tan^2(\sqrt{ab}\theta)|\nabla\alpha|^2|H|^2\Big]+o_k(1)\\
\geq\quad & \cos^{-\frac{1}{b}}(\sqrt{ab}\alpha)[(a-\frac{\tilde{b}}{b}a)\tan^2(\sqrt{ab}\theta)|\nabla\alpha|^2|H|^2\\
&+(a|\overline{\nabla}J_\Sigma|^2-\tilde{a}|H|^2)|H|^2]+o_k(1).
\end{aligned}$$

Set $b=\tilde{b}$, $a=2\tilde{a}$, by the same argument as Main Theorem 2, we get the contradiction. Thus we choose $ab=2\tilde{a}\tilde{b}>4\frac{(|T|+1)^2}{|T|^2}$. This completes the proof.

Q. E. D.

5. A finite time blow-up example

In this section, we construct a finite time blow-up example of symplectic mean curvature flows in $\mathbf{C}^2$. Let $\gamma(s)=(x(s),y(s))$ be a regular planar curve in $\mathbf{R}^2$ for $s\in(-\infty,\infty)$. We consider the surface in $\mathbf{C}^2$ defined by

$$F(s,\theta)=(x(s)e^{i\theta},y(s)e^{i\theta}):=\gamma(s)e^{i\theta},$$

where $i=\sqrt{-1}$ and $\theta\in\mathbf{R}$.

Then

$$F_s(s,\theta)=(x'(s)\cos\theta,x'(s)\sin\theta,y'(s)\cos\theta,y'(s)\sin\theta),$$

$$F_\theta(s,\theta)=(-x(s)\sin\theta,x(s)\cos\theta,-y(s)\sin\theta,y(s)\cos\theta).$$

The normal vectors are

$$\nu_s(s,\theta) = (-y'(s)\cos\theta, -y'(s)\sin\theta, x'(s)\cos\theta, x'(s)\sin\theta),$$
$$\nu_\theta(s,\theta) = (y(s)\sin\theta, -y(s)\cos\theta, -x(s)\sin\theta, x(s)\cos\theta).$$

The induced metric is

$$g_{ss}(s,\theta) = (x'(s))^2 + (y'(s))^2,\ g_{\theta\theta}(s,\theta) = x^2(s) + y^2(s),\ g_{s\theta} = 0.$$

One checks that the Kähler angle is

$$\cos\alpha = \frac{(x(s)x'(s) + y(s)y'(s))}{\sqrt{(x^2(s)+y^2(s))((x'(s))^2+(y'(s))^2)}}.$$

The second fundamental form is

$$h^s_{ss}(s,\theta) = x'(s)y''(s) - x''(s)y'(s),\ h^s_{\theta\theta} = x(s)y'(s) - x'(s)y(s),\ h^s_{s\theta} = h^\theta_{s\theta} = h^\theta_{\theta\theta} = 0,$$

and the mean curvature vector is

$$H = \frac{1}{\sqrt{(x'(s))^2+(y'(s))^2}}\left(\frac{x'(s)y''(s) - y'(s)x''(s)}{(x'(s))^2+(y'(s))^2} + \frac{x(s)y'(s) - x'(s)y(s)}{x^2(s)+y^2(s)}\right)n(s)e^{i\theta},$$

where $n(s)$ is the unit normal vector of $\gamma(s)$, and

$$n(s)e^{i\theta} = \frac{1}{\sqrt{(x'(s))^2+(y'(s))^2}}\nu_s.$$

These computations clearly yield the following proposition.

Proposition 5.1. *Let $r(s) = \sqrt{x^2(s)+y^2(s)}$. If the regular planar curve $\gamma(s) = (x(s), y(s))$ satisfies that $r'(s) > 0$, then the surface in $\mathbf{C}^2$ defined by the map $F(s,\theta) = (x(s)e^{i\theta}, y(s)e^{i\theta})$ for $s,\theta \in \mathbf{R}$ is symplectic. If $\gamma(t,s) = (x(t,s), y(t,s))$ satisfies the curvature flow equation*

$$\frac{d}{dt}\gamma(t,s) = \frac{1}{\sqrt{(x')^2+(y')^2}}\left(\frac{x'y''-y'x''}{(x')^2+(y')^2} + \frac{xy'-x'y}{x^2+y^2}\right)n(t,s), \tag{5.1}$$

where $()' = d/ds$, $()'' = d^2/ds^2$, and $n(t,s)$ is the unit normal vector of $\gamma(t,s)$, then $F(s,\theta) = (x(t,s)e^{i\theta}, y(t,s)e^{i\theta})$ satisfies the mean curvature flow equation.

The curve flow (5.1) first arises in the construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbf{C}^n$ (c.f. [1], [2], [10], [17], [18]).

The equation (5.1) can also be written as

$$\frac{d}{dt}\gamma(t,s) = k(s,t) - \frac{\gamma^\perp(s,t)}{|\gamma(s,t)|^2}, \tag{5.2}$$

where k is the curvature of γ and $\gamma^\perp$ is the projection of the vector γ on the orthogonal complement of the tangent direction of γ.

Theorem 5.2. *Let $\gamma_0(s) = \log(1+s)e^{is}$, $s \in [0,\infty)$. then the curve flow (5.1) with initial data γ_0 blows up at a finite time.*

Proof: Let $r_0(s) = \log(1+s)$. Then $\gamma_0(s) = r_0(s)e^{is}$. It is proved in [17] (Lemma 4.3) that the curve flow takes the form

$$\gamma(t,s) = r(t,s)e^{is}.$$

Computing directly, one sees that the equation (5.1) is deduced to the following form:

$$\frac{d}{dt}\gamma(t,s) = \frac{3(r')^2 - rr'' + 2r^2}{(r^2 + (r')^2)^2}(-y', x'),$$

because

$$\frac{d}{dt}\gamma(t,s) = ((\frac{d}{dt}r(t,s))\cos s, (\frac{d}{dt}r(t,s))\sin s),$$

the radius function $r(t,s)$ satisfies the equation

$$\frac{d}{dt}r(t,s) = \frac{rr'' - 3(r')^2 - 2r^2}{r^3 + r(r')^2}.$$

We consider the weighted area $A(t)$ of the triangular region

$$\{ue^{is} \mid 0 \le s < \infty,\ 0 \le u \le r(t,s)\},$$

$$A(t) = \frac{1}{2}\int_0^\infty \frac{r^2(t,s)}{(1+s)^2}ds.$$

We have

$$\begin{aligned}
\frac{d}{dt}A(t) &= \int_0^\infty r(t,s)\frac{d}{dt}r(t,s)\frac{1}{(1+s)^2}ds \\
&= \int_0^\infty \frac{1}{(1+s)^2}\frac{rdr'}{r^2+(r')^2} - \int_0^\infty \frac{1}{(1+s)^2}\frac{(r')^2}{r^2+(r')^2}ds - 2 \\
&= \int_0^\infty \frac{1}{(1+s)^2}\frac{d(r'/r)}{1+(r'/r)^2} - 2 \\
&\le 2\int_0^\infty \frac{1}{(1+s)^3}\arctan(r'/r)ds - 2 \\
&\le \frac{\pi}{2} - 2 < 0.
\end{aligned}$$

Therefore the curve flow must blow up at a finite time.

Q. E. D.

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